

# Supersymmetric Rényi Entropy and Weyl Anomalies in Six-Dimensional $(2,0)$ Theories

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**ABSTRACT:** We propose a closed formula of the universal part of supersymmetric Rényi entropy  $S_q$  for  $(2,0)$  superconformal theories in six-dimensions. We show that  $S_q$  across a spherical entangling surface is a cubic polynomial of  $\gamma := 1/q$ , with all coefficients expressed in terms of the newly discovered Weyl anomalies  $a$  and  $c$ . This is equivalent to a similar statement of the supersymmetric free energy on conic (or squashed) six-sphere. We first obtain the closed formula by promoting the free tensor multiplet result and then provide an independent derivation by assuming that  $S_q$  can be written as a linear combination of 't Hooft anomaly coefficients. We discuss a possible lower bound  $\frac{a}{c} \geq \frac{3}{7}$  implied by our result.

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# 1. Introduction

Exact results in interacting quantum field theories are rare. Even less is known about the six-dimensional  $(2,0)$  theories, although they are the local conformal field theories (CFTs) with maximal supersymmetry in the maximum number of dimensions [1, 2], which actually play important roles in understanding lower dimensional supersymmetric physics [3–7]. The main obstacle is that the proper formulation of the interacting theories is still lacking, for instance in the path integral formalism.<sup>1</sup> This also makes it challenging to study the theories in curved spaces. In particular it is unclear how to perform the supersymmetric localization [18–20] directly.

Recently alternative approaches to  $6d$   $(2,0)$  theories, such as effective actions on the moduli space and the superconformal bootstrap, are advocated in [21, 22] and in [23, 24], respectively. In particular, the Weyl anomaly coefficients  $a_{\mathfrak{g}}$  and  $c_{\mathfrak{g}}$  have been determined for the  $(2,0)$  superconformal field theory (SCFT) characterized by a Lie algebra  $\mathfrak{g}$ ,<sup>2</sup>

$$\bar{a}_{\mathfrak{g}} := \frac{a_{\mathfrak{g}}}{a_{\mathfrak{u}(1)}} = \frac{16}{7} h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}} , \quad \bar{c}_{\mathfrak{g}} := \frac{c_{\mathfrak{g}}}{c_{\mathfrak{u}(1)}} = 4 h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}} , \quad (1.1)$$

where  $r_{\mathfrak{g}}$ ,  $d_{\mathfrak{g}}$  and  $h_{\mathfrak{g}}^{\vee}$  are the rank, dimension and dual Coxeter number of the compact simply-laced Lie algebra  $\mathfrak{g}$ , respectively.  $a$  and  $c$  appear generally as coefficients of the anomalous trace of the stress tensor in a six-dimensional curved background [25, 26],

$$\langle T_{\mu}^{\mu} \rangle \sim a E_6 + \sum_{i=1}^3 c_i I_i , \quad (1.2)$$

where  $E_6$  is the Euler density while  $I_i$  are Weyl invariants. In the presence of  $(2,0)$  superconformal symmetry,  $c_{i=1,2,3}$  are proportional to a single coefficient  $c$ . One interesting fact is that both  $\bar{a}_{\mathfrak{g}}$  and  $\bar{c}_{\mathfrak{g}}$  will be uniquely fixed once we assume that they are linear combinations of the 't Hooft anomaly coefficients,  $h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}}$  and  $r_{\mathfrak{g}}$ . This can be done by combining the large  $N$  values (from holography [27–30]) and the free tensor multiplet values [31, 32].

As robust observables, the 't Hooft anomalies of the continuous global symmetries in  $6d$   $(2,0)$  theories have been worked out [33–39]. They are organized in an 8-form anomaly polynomial,

$$\mathcal{I}_8 = h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} \frac{p_2(R)}{24} + r_{\mathfrak{g}} \mathcal{I}_{\mathfrak{u}(1)} , \quad (1.3)$$

where  $p_2(R)$  is the second Pontryagin class of the field strength of the  $SO(5)$  R-symmetry background and  $\mathcal{I}_{\mathfrak{u}(1)}$  is the anomaly polynomial of a free Abelian tensor multiplet.

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<sup>1</sup>For the attempts to write down a Lagrangian, see for instance [8–12] and for other field theoretical attempts, see [13–17].

<sup>2</sup> $\mathfrak{g} = \mathfrak{u}(1)$  corresponds to a free Abelian tensor multiplet.

As in other even dimensions, it is known that  $a_{\mathfrak{g}}$  determines both the universal part<sup>3</sup> of the sphere partition function and the universal entanglement entropy associated with a spherical entangling surface (in flat space) [40]. On the other hand, it was pointed out that  $c_{\mathfrak{g}}$  determines both the 2-point and the 3-point functions of the stress tensor in the vacuum in flat space [23, 24]. Due to the intrinsic relations between the flat space stress tensor correlators and the nearly-round sphere partition function, it is therefore attempting to ask whether one can fully determine the partition function on a branched ( $q$ -deformed) sphere,<sup>4</sup> which is directly related to the supersymmetric Rényi entropy  $S_q$ .

The concept supersymmetric Rényi entropy was first introduced in three-dimensions [41–43], and later studied in four-dimensions [44, 45, 47], five-dimensions [48, 49] and for free tensor multiplets in six-dimensions [50].<sup>5</sup> By turning on certain R-symmetry background fields (chemical potentials), one can calculate the partition function  $Z_q$  on a  $q$ -branched sphere  $\mathbb{S}_q^d$ , and define the supersymmetric Rényi entropy as

$$S_q = \frac{1}{1-q} [\log Z_q(\mu(q)) - q \log Z_1(0)] , \quad (1.4)$$

which is a supersymmetric refinement of the ordinary Rényi entropy (which is non-supersymmetric because of the conical singularity).<sup>6</sup> The quantities defined in (1.4) are UV divergent in general but one can extract universal parts free of ambiguities. For instance, for  $\mathcal{N} = 4$  SYM in four-dimensions, the log coefficient of  $S_q$  as a function of  $q$  and three chemical potentials  $\mu_1, \mu_2, \mu_3$  (corresponding to three independent R-symmetry Cartans) has been shown to be protected from the interactions [44]. It also receives a precise check from the holographic computation on the  $5d$  BPS STU topological black holes [44]. Furthermore, there are universal relations between the Weyl anomaly coefficients  $a, c$  and the supersymmetric Rényi entropy in  $4d$   $\mathcal{N} = 1, 2$  SCFTs, which provides a new way to understand the Hofman-Maldacena bounds [45].<sup>7</sup> The above facts indicate that the supersymmetric Rényi entropy may be used as a new robust observable to understand SCFTs.

In this work we show that the supersymmetric Rényi entropy of  $6d$   $(2, 0)$  SCFTs characterized by simply-laced Lie algebra  $\mathfrak{g}$  is given by a cubic polynomial of  $\gamma := \frac{1}{q}$

$$S_\gamma^{(2,0)} = \sum_{n=0}^3 s_n (\gamma - 1)^n , \quad (1.5)$$

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<sup>3</sup>By “universal” we mean scheme-independent.

<sup>4</sup>A branched sphere is a sphere with a conical singularity with the deformation parameter  $q - 1$ .

<sup>5</sup>The supersymmetric Rényi entropy was recently studied in two-dimensional  $(2, 2)$  SCFTs [51] in a slightly different way.

<sup>6</sup>For CFTs, the Rényi entropy (or supersymmetric one) associated with a spherical entangling surface in flat space can be mapped to that on a sphere. Throughout this work we take the “regularized cone” boundary conditions.

<sup>7</sup>Some of  $a/c$  bounds by Hofman and Maldacena [46] coincide with Rényi entropy inequalities.

with four coefficients

$$s_0 = \frac{7}{12} \bar{a}_{\mathfrak{g}} , \quad s_1 = \frac{1+2r_1r_2}{12} \bar{c}_{\mathfrak{g}} , \quad s_2 = \frac{r_1r_2}{12} \bar{c}_{\mathfrak{g}} , \quad s_3 = \frac{r_1^2r_2^2}{12} \frac{7\bar{a}_{\mathfrak{g}} - 3\bar{c}_{\mathfrak{g}}}{4} , \quad (1.6)$$

where  $r_1$  and  $r_2$  are background parameters denoting the weights of the two  $U(1)$  chemical potentials associated to the two R-symmetry Cartans, satisfying the supersymmetry constraint  $r_1 + r_2 = 1$ .<sup>8</sup> Since both  $\bar{a}_{\mathfrak{g}}$  and  $\bar{c}_{\mathfrak{g}}$  are linear combinations of the 't Hooft anomaly coefficients, one may rewrite the closed formula  $S_{\gamma}^{(2,0)}$  (1.5) also as a linear combination of  $h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}}$  and  $r_{\mathfrak{g}}$ ,

$$S_{\gamma}^{(2,0)} = h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} H_{\gamma} + r_{\mathfrak{g}} T_{\gamma} , \quad (1.7)$$

with coefficients as cubic polynomials of  $\gamma$

$$T_{\gamma} = \frac{r_1^2r_2^2}{12}(\gamma-1)^3 + \frac{r_1r_2}{12}(\gamma-1)^2 + \frac{1+2r_1r_2}{12}(\gamma-1) + \frac{7}{12} , \quad (1.8)$$

$$H_{\gamma} = \frac{r_1^2r_2^2}{12}(\gamma-1)^3 + \frac{r_1r_2}{3}(\gamma-1)^2 + \frac{1+2r_1r_2}{3}(\gamma-1) + \frac{4}{3} . \quad (1.9)$$

We derive the closed formula (1.5) by promoting the supersymmetric Rényi entropy of a free tensor multiplet. The free tensor multiplet result is nothing but  $T_{\gamma}$  in the alternative expression (1.7), which can be directly computed using heat kernel method. Demanding that entanglement entropy  $S_{\gamma=1}$  is proportional to  $a_{\mathfrak{g}}$  and both the first and the second  $\gamma$ -derivatives at  $\gamma = 1$  are proportional to  $c_{\mathfrak{g}}$ , we could fix the coefficients  $s_0, s_1$  and  $s_2$  in (1.5). We fix the remaining  $s_3$  by demonstrating a precise relation between the large  $\gamma$  (small  $q$ ) behavior of supersymmetric Rényi entropy and the supersymmetric Casimir energy on extremely squashed sphere.<sup>9</sup> As a nontrivial test of our result (1.5), we show that  $H_{\gamma}$  precisely agrees with the holographic result computed from the BPS topological black hole in  $7d$  gauged supergravity.

The fact that  $T_{\gamma}$  and  $H_{\gamma}$  are precisely the free multiplet result and the holographic result, respectively, actually provides an alternative derivation of the closed formula (1.5). Consider  $T_{\gamma}$  and  $H_{\gamma}$  as independent results from the free field computation and the holographic dual, respectively, one can uniquely determine the closed formula of  $S_{\gamma}^{(2,0)}$  by assuming that  $S_{\gamma}^{(2,0)}$  is a linear combination of the 't Hooft anomaly coefficients. This assumption can be reasonably imposed once we are aware of any two of  $s_{n=0,1,2,3}$  as linear combinations of  $h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}}$  and  $r_{\mathfrak{g}}$ .

This paper is organized as follows. We begin with the general study of the relations between the perturbative supersymmetric Rényi entropy around  $q = 1$  and the integrated correlation functions (stress tensor and R-current) in Section 2, which works for general dimensions. We focus on the first and the second derivative at  $q = 1$ . Then we review the supersymmetric Rényi entropy of free tensor multiplets in

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<sup>8</sup>We only consider non-negative weights of the chemical potentials,  $r_1 \geq 0$  and  $r_2 \geq 0$ .

<sup>9</sup>This relation was first advertised in [45] in four-dimensions.

Section 3. Built up on these facts, we propose a way to determine the supersymmetric Rényi entropy for interacting  $(2,0)$  theories in Section 4. In Section 5, we show a general relation between the  $q \rightarrow 0$  behavior of supersymmetric Rényi entropy and supersymmetric Casimir energy, which is used to determine the remaining unfixed coefficient in the proposed formula in the previous section. Finally we give a precise test of our results by comparing with the holographic results in Section 6.

## 2. Near $q = 1$ expansion

We begin with the perturbative expansion of supersymmetric Rényi entropy (associated with spherical entangling surface) around  $q = 1$ . This can be considered as an extension of the previous study of the ordinary Rényi entropy near  $q = 1$ . Although our main concern will be  $6d$   $(2,0)$  SCFTs, we keep the discussions in this section valid for any SCFT with conserved R-symmetries in  $d$ -dimensions.

Following the way in [52, 53]<sup>10</sup>, we consider the supersymmetric partition function on  $\mathbb{S}_q^1 \times \mathbb{H}^{d-1}$  with background gauge fields (R-symmetry chemical potentials), which can be used to compute the supersymmetric Rényi entropy across a spherical entangling surface, see  $\mathbb{S}^{d-2}$ , in flat space. We work in grand canonical ensemble. The partition function on  $\mathbb{S}_{\beta=2\pi q}^1 \times \mathbb{H}^{d-1}$  can be written as

$$Z[\beta, \mu] = \text{Tr} \left( e^{-\beta(\hat{E} - \mu \hat{Q})} \right). \quad (2.1)$$

The state variables can be computed as follows

$$E = \left( \frac{\partial I}{\partial \beta} \right)_\mu - \frac{\mu}{\beta} \left( \frac{\partial I}{\partial \mu} \right)_\beta, \quad (2.2)$$

$$S = \beta \left( \frac{\partial I}{\partial \beta} \right)_\mu - I, \quad (2.3)$$

$$Q = -\frac{1}{\beta} \left( \frac{\partial I}{\partial \mu} \right)_\beta, \quad (2.4)$$

where  $I := -\log Z$ . Therefore we get energy expectation value by (2.2)

$$E = \frac{\text{Tr} \left( e^{-\beta(\hat{E} - \mu \hat{Q})} \hat{E} \right)}{\text{Tr} \left( e^{-\beta(\hat{E} - \mu \hat{Q})} \right)}, \quad (2.5)$$

and the charge expectation value by (2.4)

$$Q = \frac{\text{Tr} \left( e^{-\beta(\hat{E} - \mu \hat{Q})} \hat{Q} \right)}{\text{Tr} \left( e^{-\beta(\hat{E} - \mu \hat{Q})} \right)}. \quad (2.6)$$

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<sup>10</sup>See [54] from the viewpoint of twisted operator.

In the presence of supersymmetry, both inverse temperature  $\beta$  and chemical potential  $\mu$  are functions of a single variable  $q$  therefore  $I$  is considered as

$$I_q := I[\beta(q), \mu(q)] . \quad (2.7)$$

The supersymmetric Rényi entropy is defined as

$$S_q = \frac{qI_1 - I_q}{1 - q} . \quad (2.8)$$

Consider the Taylor expansion around  $q = 1$ , with  $\delta q = q - 1$ ,

$$S_q = S_{\text{EE}} + \sum_{n=2}^{\infty} \frac{1}{n!} \left( \frac{\partial^n I_q}{\partial q^n} \right)_{q=1} \delta q^{n-1} . \quad (2.9)$$

## 2.1 $\partial_q I_q$

We will first consider  $\partial_q I_q$ . The first derivative with respect to  $q$  can be written as

$$\frac{dI_q}{dq} = \left( \frac{\partial I}{\partial \beta} \right)_{\mu} \beta'(q) + \left( \frac{\partial I}{\partial \mu} \right)_{\beta} \mu'(q) . \quad (2.10)$$

Using (2.2) and (2.4), we can rewrite it as

$$\frac{dI_q}{dq} = (E - \mu Q) \beta'(q) - \beta Q \mu'(q) . \quad (2.11)$$

The  $q$ -dependence of the temperature and the chemical potential can be read off from the supersymmetric background (including metric and R-symmetry gauge field),

$$\beta(q) = 2\pi q , \quad \mu(q) = \alpha \frac{q-1}{q} , \quad (2.12)$$

where  $\beta(q)$  is determined by the geometric fact and  $\mu(q)$  is solved from the Killing spinor equation on the background.  $\alpha$  is some number which may be different in various rigid supersymmetric backgrounds.<sup>11</sup> The first  $q$ -derivative of  $I_q$  is simplified by using (2.12)

$$I'_q = 2\pi(E - \alpha Q) . \quad (2.13)$$

Notice that in general both  $E$  and  $Q$  are functions of  $q$ . Also  $E$  and  $Q$  here are expectation values rather than operators.

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<sup>11</sup>In the case of multiple chemical potentials, one should use  $\alpha_{i=1,2,\dots,R}$ , where  $R$  denotes the number of  $U(1)$  R-symmetry Cartans.  $i$  should be summed over for  $\alpha_i Q^i$ .

## 2.2 $S'_{q=1}$ and $I''_{q=1}$

From (2.9) we see that

$$S'_{q=1} = \frac{1}{2} I''_{q=1} . \quad (2.14)$$

Let us take one more derivative above on the first derivative (2.13) and take use of (2.5) and (2.6)

$$I''_q = -4\pi^2 \left( \frac{\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})} (\hat{E} - \alpha\hat{Q})^2)}{\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})})} - \frac{[\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})} (\hat{E} - \alpha\hat{Q}))]^2}{[\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})})]^2} \right) , \quad (2.15)$$

which can be simplified in the limit  $q \rightarrow 1$  by using  $\mu = 0$  at  $q = 1$

$$S'_{q=1} = -2\pi^2 \left( \frac{\text{Tr}(e^{-\beta\hat{E}} (\hat{E} - \alpha\hat{Q})^2)}{\text{Tr}(e^{-\beta\hat{E}})} - \frac{[\text{Tr}(e^{-\beta\hat{E}} (\hat{E} - \alpha\hat{Q}))]^2}{[\text{Tr}(e^{-\beta\hat{E}})]^2} \right)_{q=1} . \quad (2.16)$$

This can be rewritten as connected correlators

$$S'_{q=1} = -2\pi^2 [\langle \hat{E}\hat{E} \rangle^c + \alpha^2 \langle \hat{Q}\hat{Q} \rangle^c - 2\alpha \langle \hat{E}\hat{Q} \rangle^c]_{\mathbb{S}_{q=1}^1 \times \mathbb{H}^{d-1}} , \quad (2.17)$$

where we have used  $[\hat{E}, \hat{Q}] = 0$  to flip the order of  $\hat{E}$  and  $\hat{Q}$ . Given that  $\langle \hat{E}\hat{Q} \rangle^c = 0$  and  $\langle \hat{E}\hat{E} \rangle^c$  has been computed in [52], we get

$$S'_{q=1} = -V_{d-1} \frac{\pi^{d/2+1} \Gamma(d/2)(d-1)}{(d+1)!} C_T - 2\pi^2 \alpha^2 \int_{\mathbb{H}^{d-1}} \int_{\mathbb{H}^{d-1}} \langle J_\tau(x) J_\tau(y) \rangle_{q=1}^c . \quad (2.18)$$

$C_T$  is defined in the flat space correlator

$$\langle T_{ab}(x) T_{cd}(0) \rangle = \frac{C_T}{x^{2d}} I_{ab,cd}(x) , \quad (2.19)$$

where

$$I_{ab,cd}(x) = \frac{1}{2} (I_{ac}(x) I_{bd}(x) + I_{ad}(x) I_{bc}(x)) - \frac{1}{d} \delta_{ab} \delta_{cd} , \quad I_{ab}(x) = \delta_{ab} - 2 \frac{x_a x_b}{x^2} . \quad (2.20)$$

Now the task is to compute the second term in (2.18). Following the way of computing  $\langle TT \rangle$  on the hyperbolic space  $\mathbb{S}_{q=1}^1 \times \mathbb{H}^{d-1}$ , one can take use of the flat space correlators in the CFT vacuum. The result is <sup>12</sup>

$$\langle \hat{Q}\hat{Q} \rangle^c = -\frac{\pi^{\frac{d-1}{2}} V_{d-1}}{2^{d-2} (d-1) \Gamma(\frac{d-1}{2})} C_v , \quad (2.21)$$

where  $C_v$  is defined in the current correlator in flat space

$$\langle J_a(x) J_b(0) \rangle = \frac{C_v}{x^{2(d-1)}} I_{ab}(x) . \quad (2.22)$$

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<sup>12</sup> $\langle J\hat{Q} \rangle$  was first computed in [55].



Then our final result of  $S'_{q=1}$  becomes

$$S'_{q=1} = -V_{d-1} \left( \frac{\pi^{\frac{d}{2}+1} \Gamma(\frac{d}{2})(d-1)}{(d+1)!} C_T - \alpha^2 \frac{\pi^{\frac{d+3}{2}}}{2^{d-3}(d-1)\Gamma(\frac{d-1}{2})} C_v \right), \quad (2.23)$$

which tells us that the first  $q$ -derivative of supersymmetric Rényi entropy at  $q = 1$  is given by a linear combination of  $C_T$  and  $C_v$ .<sup>13</sup> This is intuitively expected because in the presence of supersymmetry, taking the derivative with respect to  $q$  is equivalent to taking the derivative with respect to  $g_{\tau\tau}$  and  $A_\tau$  in the same time.<sup>14</sup>  $q$ -deformation can be often equivalent to the squashing  $b := \sqrt{q}$ , therefore this formula also shows the relation between  $\partial_{b=1}^2$  of the free energy on squashed sphere and flat space correlators. It is clear from the above derivation that this formula works both for free theories and interacting SCFTs in general  $d$ -dimensions. In the particular case of  $6d$   $(2,0)$  SCFTs, the 2-point function of the stress tensor is determined by the central charge  $c_g$  in (1.1) [23, 24]. Therefore the integrated 2-point function is proportional to  $c_g$ . Furthermore,  $S'_{q=1}$  is also proportional to  $c_g$ , because the stress tensor and the R-current in the right hand side of (2.23) live in the same supermultiplet.<sup>15</sup> The same thing happens in  $\mathcal{N} = 4$  SYM [44].

### 2.3 $S''_{q=1}$ and $I'''_{q=1}$

From (2.9) we see that

$$S''_{q=1} = \frac{1}{6} I'''_{q=1}. \quad (2.24)$$

One may go straightforward to compute  $I'''_q$  by taking one more derivative above on (2.15)

$$\begin{aligned} \frac{I'''_q}{8\pi^3} &= \frac{\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})}(\hat{E}-\alpha\hat{Q})^3)}{\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})})} - 3 \frac{\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})}(\hat{E}-\alpha\hat{Q})^2) \text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})}(\hat{E}-\alpha\hat{Q}))}{[\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})})]^2} \\ &\quad + 2 \frac{[\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})}(\hat{E}-\alpha\hat{Q}))]^3}{[\text{Tr}(e^{-\beta(\hat{E}-\mu\hat{Q})})]^3}, \end{aligned} \quad (2.25)$$

which may be simplified at  $q = 1$  where  $\mu = 0$

$$\begin{aligned} \frac{I'''_{q=1}}{8\pi^3} &= \left( \frac{\text{Tr}(e^{-\beta\hat{E}}(\hat{E}-\alpha\hat{Q})^3)}{\text{Tr}e^{-\beta\hat{E}}} - 3 \frac{\text{Tr}(e^{-\beta\hat{E}}(\hat{E}-\alpha\hat{Q})^2) \text{Tr}(e^{-\beta\hat{E}}(\hat{E}-\alpha\hat{Q}))}{[\text{Tr}e^{-\beta\hat{E}}]^2} \right. \\ &\quad \left. + 2 \frac{[\text{Tr}(e^{-\beta\hat{E}}(\hat{E}-\alpha\hat{Q}))]^3}{[\text{Tr}e^{-\beta\hat{E}}]^3} \right)_{q=1}. \end{aligned} \quad (2.26)$$

<sup>13</sup>In another word, a linear combination of the integrated stress tensor 2-point function and the integrated R-current 2-point function.

<sup>14</sup>This was first suggested in [44].

<sup>15</sup>For  $(2,0)$  tensor multiplet, this supermultiplet was studied explicitly in [56].

This can be further written in terms of connected correlation functions,

$$S''_{q=1} = \frac{1}{6} I'''_{q=1} = \frac{4\pi^3}{3} [\langle \hat{E} \hat{E} \hat{E} \rangle^c - \alpha^3 \langle \hat{Q} \hat{Q} \hat{Q} \rangle^c - 3\alpha \langle \hat{E} \hat{E} \hat{Q} \rangle^c + 3\alpha^2 \langle \hat{E} \hat{Q} \hat{Q} \rangle^c]_{\mathbb{S}^1_{q=1} \times \mathbb{H}^{d-1}} , \quad (2.27)$$

where we have used  $[\hat{E}, \hat{Q}] = 0$  because  $\hat{Q}$  is conserved charge. The integrated correlators in (2.27) can be computed by transforming the corresponding flat space correlators,  $\langle TTT \rangle, \langle JJJ \rangle, \langle TTJ \rangle, \langle TJJ \rangle$  in the CFT vacuum.<sup>16</sup> These correlators in flat space can be determined up to some coefficients for general CFTs in  $d$ -dimensions by conformal Wald identities [57, 58]. In the presence of  $6d$   $(2, 0)$  superconformal symmetry, both the 2- and 3-point functions of the stress tensor supermultiplet are uniquely determined in terms of a single parameter, the central charge  $c_g$  [23, 24]. And the right hand side of (2.27) should be proportional to  $c_g$ , because the stress tensor and the R-current belong to the same supermultiplet.<sup>17</sup> The same thing can be seen in  $\mathcal{N} = 4$  SYM [44].

### 3. Abelian tensor multiplet

The six-dimensional  $(2, 0)$  superconformal algebra is  $\mathfrak{osp}(8^*|4)$ . While it is easy to identify a free Abelian tensor multiplet that realizes the  $(2, 0)$  superconformal symmetry, the existence of interacting  $(2, 0)$  theories was only inferred from decoupling limits of string constructions [62–64]. See for instance [65] for a review of various aspects of  $6d$   $(2, 0)$  theories.

Now we review the supersymmetric Rényi entropy of free tensor multiplets [50]. For free fields, the Rényi entropy associated with a spherical entangling surface in flat space can be computed by working on a hyperbolic space  $\mathbb{S}^1_\beta \times \mathbb{H}^5$  and using heat kernel method.<sup>18</sup> A six-dimensional  $(2, 0)$  tensor multiplet includes 5 real scalars, 2 Weyl fermions and a 2-form field with self-dual strength. The 2-form field with self-dual strength can be considered as a chiral 2-form field with half of the degrees of freedom.

#### 3.1 Heat kernel

The partition function of free fields on  $\mathbb{S}^1_{\beta=2\pi q} \times \mathbb{H}^5$  can be obtained by heat kernel method,<sup>19</sup>

$$\log Z(\beta) = \frac{1}{2} \int_0^\infty \frac{dt}{t} K_{\mathbb{S}^1_\beta \times \mathbb{H}^5}(t) , \quad (3.1)$$

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<sup>16</sup>We leave the explicit computations of these correlators elsewhere.

<sup>17</sup>By representation theory, the stress tensor belongs to a half BPS multiplet. In superspace, the 2-, 3- and 4-point functions of all half BPS multiplets are known to admit a unique structure [59–61].

<sup>18</sup>Six-dimensional  $(2, 0)$  theories have been studied in  $AdS_5 \times S^1$  recently in the viewpoint of rigid holography [66].

<sup>19</sup>For Rényi entropy of free fields in other higher dimensions, see for instance [67–70].

where  $K_{\mathbb{S}_\beta^1 \times \mathbb{H}^5}(t)$  is the heat kernel of the associated conformal Laplacian. The kernel can be factorized when the spacetime is a direct product,

$$K_{\mathbb{S}_\beta^1 \times \mathbb{H}^5}(t) = K_{\mathbb{S}_\beta^1}(t) K_{\mathbb{H}^5}(t) . \quad (3.2)$$

The kernel on a circle  $K_{\mathbb{S}_\beta^1}(t)$  is known to be <sup>20</sup>

$$K_{\mathbb{S}_\beta^1}(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{n \neq 0, \epsilon \mathbb{Z}} e^{-\frac{\beta^2 n^2}{4t}} . \quad (3.3)$$

In the presence of a chemical potential  $\mu$ , it is twisted to be [55]

$$\tilde{K}_{\mathbb{S}_\beta^1}(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{n \neq 0, \epsilon \mathbb{Z}} e^{-\frac{\beta^2 n^2}{4t} + i2\pi n \mu + i\pi n f} , \quad (3.4)$$

where  $f = 0$  for scalars and  $f = 1$  for fermions. Finally the kernels on the hyperbolic space  $K_{\mathbb{H}^5}(t)$  can be written as follows because  $\mathbb{H}^5$  is homogeneous,

$$K_{\mathbb{H}^5}(t) = \int d^5x \sqrt{g} K_{\mathbb{H}^5}(x, x, t) = V_5 K_{\mathbb{H}^5}(0, t) . \quad (3.5)$$

The regularized volume  $V_5 = \pi^2 \log(\ell/\epsilon)$ .  $\epsilon$  is the UV cutoff of the theory in the original space <sup>21</sup> and  $\ell$  is the curvature radius of  $\mathbb{H}^5$ . Note that the kernels  $K_{\mathbb{H}^5}(0, t)$  for free fields with different spins are known. See [50] and references there.

### 3.2 Rényi entropy

The total Rényi entropy of a tensor multiplet can be obtained by summing up the contributions of 5 real scalars, 2 Weyl fermions and a chiral 2-form,

$$S_q^{free} = 5 \times \frac{S_q^s}{2} + 2S_q^f + \frac{S_q^v}{2} , \quad (3.6)$$

where the Rényi entropy for fields with different spins can be computed by using the corresponding heat kernels. For the details of this computation we refer to [50]. We will instead list the results here. The Rényi entropy of a  $6d$  real scalar is

$$S_q^s = \frac{(q+1)(3q^2+1)(3q^2+2)}{15120q^5} \frac{V_5}{\pi^2} , \quad (3.7)$$

and the Rényi entropy of a  $6d$  Weyl fermion is

$$S_q^f = \frac{(q+1)(1221q^4+276q^2+31)}{120960q^5} \frac{V_5}{\pi^2} , \quad (3.8)$$

---

<sup>20</sup>For fermions, the boundary conditions are anti-periodic.

<sup>21</sup>This is the  $q$ -fold space with a conical singularity, which is used to compute Rényi entropy by replica trick.

and that of a  $6d$  2-from field is

$$S_q^v = \frac{(q+1)(37q^2+2) + 877q^4 + 4349q^5}{5040q^5} \frac{V_5}{\pi^2} . \quad (3.9)$$

It is worth to mention that, to get the correct Rényi entropy for the two form field, one has to take into account a  $q$ -independent constant shift due to the edge modes [50], like what should be done for the gauge field in  $4d$  [71, 72]. Finally the Rényi entropy for a free  $(2, 0)$  tensor multiplet is

$$S_q^{free} = \frac{(q+1)(28q^2+3) + 313q^4 + 1305q^5}{2880q^5} \frac{V_5}{\pi^2} . \quad (3.10)$$

It has been checked that  $\partial_{q=1}^0$ ,  $\partial_{q=1}^1$  and  $\partial_{q=1}^2$  of  $S_q^{free}$  are consistent [50] with the previous results about the tensor multiplet [31, 32, 73].

### 3.3 $S_q$ and $S_\gamma$

Before moving on, let us represent  $S_q^{free}$  in terms of

$$S_\gamma := \frac{\pi^2}{V_5} S_q , \text{ with } \gamma := 1/q ,$$

$$S_\gamma^{free} = \frac{1}{960}(\gamma-1)^5 + \frac{1}{160}(\gamma-1)^4 + \frac{7}{288}(\gamma-1)^3 + \frac{1}{18}(\gamma-1)^2 + \frac{\gamma-1}{6} + \frac{7}{12} . \quad (3.11)$$

The reason why  $S_\gamma$  is convenient is that, the series expansion near  $\gamma = 1$  has finite terms while the expansion of  $S_q$  near  $q = 1$  has infinite terms. We will use  $S_\gamma$  instead of  $S_q$  to express Rényi entropy and supersymmetric Rényi entropy from now on. It is worth to note the relations between the derivatives with respect to  $q$  and the derivatives with respect to  $\gamma$  at  $q = 1/\gamma = 1$ ,

$$\partial_\gamma S_\gamma = -\partial_q S_q \Big|_{q=1/\gamma=1} \cdot \frac{\pi^2}{V_5} , \quad \partial_\gamma^2 S_\gamma = \left( 2\partial_q S_q + \partial_q^2 S_q \right) \Big|_{q=1/\gamma=1} \cdot \frac{\pi^2}{V_5} . \quad (3.12)$$

### 3.4 Supersymmetric Rényi entropy

The supersymmetric Rényi entropy of a free tensor multiplet can be computed by the twisted kernel (3.4) on the supersymmetric background. The R-symmetry group of  $6d$   $(2, 0)$  theories is  $SO(5)$ , which has two  $U(1)$  Cartans. Therefore one can turn on two independent R-symmetry background gauge fields (chemical potentials) to twist the boundary conditions for scalars and fermions along the replica circle  $\mathbb{S}_\beta^1$ . A general analysis of the Killing spinor equation on the conic space  $(\mathbb{S}_q^6 \text{ or } \mathbb{S}_{\beta=2\pi q}^1 \times \mathbb{H}^5)$  leads to the solution of the R-symmetry chemical potential [50]<sup>22</sup>

$$\mu(q) := k_i A^i = \frac{q-1}{2} , \quad (3.13)$$

---

<sup>22</sup>The Killing spinors on round sphere have been explored in [74].

with  $k_1$  and  $k_2$  being the R-charges of the Killing spinor under the two  $U(1)$  Cartans, respectively. We choose  $k_1 = k_2 = \frac{1}{2}$  and the two background fields can be expressed as

$$A^1 = (q-1)r_1, \quad A^2 = (q-1)r_2, \quad \text{with } r_1 + r_2 = 1. \quad (3.14)$$

This is the most general background satisfying (3.13). For each component field in the tensor multiplet, one has to first figure out the Cartan charges  $k_1$  and  $k_2$  and then compute the chemical potential by  $k_1 A^1 + k_2 A^2$ . Then one can compute the free energy on  $\mathbb{S}_\beta^1 \times \mathbb{H}^5$  using the twisted heat kernel and get the supersymmetric Rényi entropy. For details, see [50].

After summing up all the component fields, the final supersymmetric Rényi entropy in terms of  $\gamma$  can be expressed as,<sup>23</sup>

$$S_\gamma^{free} = \frac{1}{12} r_1^2 r_2^2 (\gamma-1)^3 + \frac{1}{12} r_1 r_2 (\gamma-1)^2 + \frac{1}{12} (1+2r_1 r_2) (\gamma-1) + \frac{7}{12}. \quad (3.15)$$

It is worth to note that, for a single  $U(1)$  background,  $r_1 = 1, r_2 = 0$ , the result becomes

$$S_\gamma = \frac{1}{12} (\gamma+6), \quad (3.16)$$

while for two  $U(1)$  backgrounds with equal values,  $r_1 = r_2 = \frac{1}{2}$ , we have

$$S_\gamma = \frac{1}{192} (\gamma-1)^3 + \frac{1}{48} (\gamma-1)^2 + \frac{1}{8} (\gamma-1) + \frac{7}{12}. \quad (3.17)$$

## 4. Interacting (2,0) theories

Having obtained the supersymmetric Rényi entropy (3.15) for a free tensor multiplet, we now try to promote it to a general form which may work for interacting (2,0) SCFTs,

$$S_\gamma^{(2,0)} = \frac{r_1^2 r_2^2}{12} \cdot A (\gamma-1)^3 + \frac{r_1 r_2}{12} \cdot B (\gamma-1)^2 + \frac{1+2r_1 r_2}{12} \cdot C (\gamma-1) + \frac{7}{12} D, \quad (4.1)$$

where the coefficients  $A, B, C, D$  will depend on the specific theory.<sup>24</sup> The factors carrying  $r_1$  and  $r_2$  should stay the same as that appearing in the free multiplet result (3.15) because they originally come from the  $\alpha_i$  ( $\alpha_1 = r_1, \alpha_2 = r_2$ ) in (2.23)(2.27), which are background parameters independent of the specific theory. Later we will see that precisely the same factors appear in the holographic supersymmetric Rényi entropy, which confirms this fact.

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<sup>23</sup>Although the form of this expression is a series expansion, the result itself is complete.

<sup>24</sup> $S_\gamma^{(2,0)}$  should be a cubic polynomial of  $\gamma$ , which is the unique option compatible with both free field result and holographic result (as we will see). The same thing happens in  $\mathcal{N} = 4$  SYM. Here we see an essential difference between the ordinary Rényi entropy and the supersymmetric one, because the type of  $q$  scaling in the ordinary Rényi entropy is not protected [53, 75].

#### 4.1 $S_{\gamma=1}^{(2,0)}$ and $a_{\mathfrak{g}}$

We would like to first determine the coefficient  $D$  in (4.1). This can be done by using the fact that, the entanglement entropy associated with a spherical entangling surface, which is nothing but  $S_{\gamma=1}$ , is proportional to  $a$ , where  $a$  is the  $a$ -type Weyl anomaly. This is true for general CFTs in even dimensions as shown in [40]. Therefore

$$\frac{S_{\text{EE}}^{(2,0)}}{S_{\text{EE}}^{\text{free}}} = \frac{a_{\mathfrak{g}}}{a_{\text{u}(1)}} . \quad (4.2)$$

This allows us to fix

$$D = \frac{a_{\mathfrak{g}}}{a_{\text{u}(1)}} = \frac{16}{7} h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}} , \quad (4.3)$$

where we have used the  $a$ -type Weyl anomaly result in  $6d$   $(2,0)$  theories [22].

#### 4.2 $\partial S_{\gamma=1}^{(2,0)}, \partial^2 S_{\gamma=1}^{(2,0)}$ and $c_{\mathfrak{g}}$

The coefficients  $C$  and  $B$  in (4.1) are determined by the first and the second  $\gamma$ -derivatives of  $S_{\gamma}^{(2,0)}$  at  $\gamma = 1$ , respectively.  $\gamma$ -derivatives can be translated into  $q$ -derivatives. Taking  $q$ -derivatives can be equivalently considered as taking derivatives with respect to background fields, therefore  $\partial S_{\gamma=1}^{(2,0)}$  and  $\partial^2 S_{\gamma=1}^{(2,0)}$  are intrinsically related to the corresponding correlators. This has been illustrated in Section 2.

Explicitly, the first  $\gamma$ -derivative (which is minus the  $q$ -derivative at  $q = 1/\gamma = 1$ ) is determined by a linear combination of the integrated stress tensor 2-point function and the integrated R-current 2-point function. The first  $q$ -derivative at  $q = 1$  is given by the formula (2.23),

$$S'_{q=1} = -V_{d-1} \left( \frac{\pi^{\frac{d}{2}+1} \Gamma(\frac{d}{2})(d-1)}{(d+1)!} C_T - \alpha^2 \frac{\pi^{\frac{d+3}{2}}}{2^{d-3}(d-1)\Gamma(\frac{d-1}{2})} C_v \right) , \quad (4.4)$$

which works for general SCFTs with conserved R-symmetries in  $d$ -dimensions.

Similarly the second  $\gamma$ -derivative at  $\gamma = 1$  is related to  $q$ -derivatives by (3.12). The second  $q$ -derivative at  $q = 1$  is determined by a linear combination of the integrated stress tensor 3-point function, the integrated R-current 3-point function and some mixed 3-point functions. This is given explicitly by (2.27)

$$S''_{q=1} = \frac{1}{6} I'''_{q=1} = \frac{4\pi^3}{3} \left[ \langle \hat{E} \hat{E} \hat{E} \rangle^c - \alpha^3 \langle \hat{Q} \hat{Q} \hat{Q} \rangle^c - 3\alpha \langle \hat{E} \hat{E} \hat{Q} \rangle^c + 3\alpha^2 \langle \hat{E} \hat{Q} \hat{Q} \rangle^c \right]_{\mathbb{S}_{q=1}^1 \times \mathbb{H}^{d-1}} , \quad (4.5)$$

which also works for general SCFTs with conserved R-symmetries in  $d$ -dimensions.

In the particular case of  $6d$   $(2,0)$  SCFTs, all the above two- and three-point functions may be uniquely determined in terms of a single parameter, the central charge  $c_{\mathfrak{g}}$  (1.1), as discussed in Section 2. <sup>25</sup>

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<sup>25</sup>This actually explains the universal ratio  $4N^3$  between the explicit results on  $\langle TT \rangle, \langle TTT \rangle, \langle JJ \rangle, \langle JJJ \rangle$  in holography and those in free tensor multiplets [32, 76].

Due to the above facts, the straightforward idea to get  $\partial S_{\gamma=1}^{(2,0)}$  and  $\partial^2 S_{\gamma=1}^{(2,0)}$  for interacting theories is to multiply

$$\frac{c_{\mathfrak{g}}}{c_{u(1)}} = 4h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}} \quad (4.6)$$

to the free multiplet values in (3.15). This actually means we can fix

$$B = C = 4h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}} . \quad (4.7)$$

The remaining coefficient  $A$  will be fixed as

$$A = h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}} \quad (4.8)$$

in the next section by studying the asymptotic  $q := 1/\gamma \rightarrow 0$  behavior of the supersymmetric Rényi entropy. Obviously, the leading contribution in the limit  $\gamma \rightarrow \infty$  is controlled only by  $A$ .

### 4.3 A closed formula

As a summary, we can completely determine a closed formula of supersymmetric Rényi entropy for  $(2,0)$  SCFTs characterized by simply-laced Lie algebra  $\mathfrak{g}$

$$\begin{aligned} S_{\gamma}^{(2,0)} &= \frac{r_1^2 r_2^2}{12} (h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}}) (\gamma - 1)^3 + \frac{r_1 r_2}{12} (4h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}}) (\gamma - 1)^2 \\ &\quad + \frac{1 + 2r_1 r_2}{12} (4h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + r_{\mathfrak{g}}) (\gamma - 1) + \left( \frac{4h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}}}{3} + \frac{7r_{\mathfrak{g}}}{12} \right) , \end{aligned} \quad (4.9)$$

$$= \frac{r_1^2 r_2^2}{48} (7\bar{a}_{\mathfrak{g}} - 3\bar{c}_{\mathfrak{g}}) (\gamma - 1)^3 + \frac{r_1 r_2}{12} \bar{c}_{\mathfrak{g}} (\gamma - 1)^2 + \frac{1 + 2r_1 r_2}{12} \bar{c}_{\mathfrak{g}} (\gamma - 1) + \frac{7}{12} \bar{a}_{\mathfrak{g}} , \quad (4.10)$$

where in the last line we have used the normalized Weyl anomalies defined in (1.1).

For a single  $U(1)$  chemical potential,

$$r_1 = 1 , \quad r_2 = 0 , \quad (4.11)$$

the result is simplified to be

$$S_{\gamma}^{(2,0)} = \frac{1}{12} \bar{c}_{\mathfrak{g}} (\gamma - 1) + \frac{7}{12} \bar{a}_{\mathfrak{g}} , \quad (4.12)$$

$$= h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} \left( \frac{1}{3} \gamma + 1 \right) + r_{\mathfrak{g}} \frac{(\gamma + 6)}{12} . \quad (4.13)$$

As for two  $U(1)$  chemical potentials with equal values,

$$r_1 = r_2 = \frac{1}{2} , \quad (4.14)$$

the result is simplified to be

$$S_{\gamma}^{(2,0)} = \frac{1}{192 \times 4} (7\bar{a}_{\mathfrak{g}} - 3\bar{c}_{\mathfrak{g}}) (\gamma - 1)^3 + \frac{1}{48} \bar{c}_{\mathfrak{g}} (\gamma - 1)^2 + \frac{1}{8} \bar{c}_{\mathfrak{g}} (\gamma - 1) + \frac{7}{12} \bar{a}_{\mathfrak{g}} , \quad (4.15)$$

$$= \frac{175 + 67\gamma + 13\gamma^2 + \gamma^3}{192} h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} + \frac{91 + 19\gamma + \gamma^2 + \gamma^3}{192} r_{\mathfrak{g}} . \quad (4.16)$$

## 5. $q \rightarrow 0$ asymptotics

In this section we discuss the  $q \rightarrow 0$  limit ( $\gamma \rightarrow \infty$ ) of supersymmetric Rényi entropy  $S_q$ . Recall the definition of  $S_q$

$$S_q = \frac{qI_1 - I_q}{1 - q} . \quad (5.1)$$

Assuming that in the limit  $q \rightarrow 0$  the free energy behaves

$$I_q = I_{(0)} q^{-\alpha} + \dots , \quad (5.2)$$

where  $\alpha \geq 0$ , one can easily get

$$S_{q \rightarrow 0} = -I_{q \rightarrow 0} \quad (5.3)$$

in the leading order. This relation does not depend on which geometric background we are working on.

The idea is that,  $\mathbb{S}_q^d$  can be conformally mapped to  $\mathbb{H}^1 \times \mathbb{S}_q^{d-1}$ , therefore the Rényi entropy (or supersymmetric) is invariant [40]. In the case with supersymmetry, one has to make sure that in the limit  $q \rightarrow 0$ , the background field on  $\mathbb{S}_q^d$  coincides with that on  $\mathbb{H}^1 \times \mathbb{S}_q^{d-1}$ . If that is the case, the asymptotic supersymmetric Rényi entropy  $S_{q \rightarrow 0}$  on  $\mathbb{S}_q^d$  will coincide with the minus free energy on  $\mathbb{H}^1 \times \mathbb{S}_{q \rightarrow 0}^{d-1}$ . The latter is determined by the supersymmetric Casimir energy [77]. We will illustrate the details in the following.

### 5.1 From $\mathbb{S}_q^d$ to $\mathbb{H}^{d-p} \times \mathbb{S}_q^p$

We start with the conformal transformation from conic sphere  $\mathbb{S}_q^d$  to hyperbolic space  $\mathbb{H}^{d-p} \times \mathbb{S}_q^p$ . Of course  $\mathbb{S}_q^d$  can be considered as the special case of  $p = d$ .

In the particular case  $p = 1$ , the transformation is nothing but the Weyl transformation discussed in [40], which offers a convenient way to compute Rényi entropy of CFTs. In this case, the branched  $d$ -sphere is described as <sup>26</sup>

$$ds^2 = \sin^2 \theta q^2 d\tau^2 + d\theta^2 + \cos^2 \theta d^2 \Omega_{d-2} , \quad (5.4)$$

with domains of coordinates given by

$$\tau \in [0, 2\pi) , \quad \theta \in \left[0, \frac{\pi}{2}\right] , \quad (5.5)$$

and  $\Omega_{d-2}$  is a standard  $d-2$ -dimensional round sphere. The metric (5.4) can be written as

$$ds^2 = \sin^2 \theta \left( q^2 d\tau^2 + \frac{1}{\sin^2 \theta} d\theta^2 + \cot^2 \theta d^2 \Omega_{d-2} \right) , \quad (5.6)$$

which can be related to the following space by dropping an overall factor  $\sin^2 \theta$  and using a coordinate transformation  $\cot \theta = \sinh \eta$

$$ds^2 = q^2 d\tau^2 + d\eta^2 + \sinh^2 \eta d^2 \Omega_{d-2} , \quad (5.7)$$

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<sup>26</sup>We normalize the radius as unit.



where  $\eta \in [0, +\infty)$ . This is the space of  $\mathbb{H}^{d-1} \times \mathbb{S}_q^1$ , which indeed fits the case of  $p = 1$ .

Now we consider the general cases,  $1 \leq p < d$ . The key observation is that, the branched sphere can be presented in different forms. For instance, we can represent  $\mathbb{S}_q^d$  as

$$ds^2 = \sin^2 \theta (d\chi^2 + \sin^2 \chi q^2 d\tau^2) + d\theta^2 + \cos^2 \theta d^2 \Omega_{d-3} , \quad (5.8)$$

with domains

$$\chi \in [0, \pi] , \quad \tau \in [0, 2\pi) , \quad \theta \in \left[0, \frac{\pi}{2}\right] , \quad (5.9)$$

and  $\Omega_{d-3}$  is a standard  $d-3$ -dimensional round sphere. Again by dropping an overall factor  $\sin^2 \theta$  and using a coordinate transformation  $\cot \theta = \sinh \eta$  for the metric (5.8), one obtains

$$ds^2 = d\chi^2 + \sin^2 \chi q^2 d\tau^2 + d\eta^2 + \sinh^2 \eta d^2 \Omega_{d-3} , \quad (5.10)$$

which is the space  $\mathbb{H}^{d-2} \times \mathbb{S}_q^2$  with  $p = 2$ . One can follow the same way to eventually figure out the Weyl transformations between  $\mathbb{S}_q^d$  and  $\mathbb{H}^{d-p} \times \mathbb{S}_q^p$  for any integer  $1 \leq p < d$ .

Since the Rényi entropy on  $\mathbb{S}_q^d$  can not depend on which particular circle we choose to create the conical singularity, one eventually arrives at the conclusion by employing the same argument in [40]:<sup>27</sup>

*The universal part of  $CFT_d$  Rényi entropy is invariant on  $\mathbb{H}^{d-p} \times \mathbb{S}_q^p$  for different integer  $p$ , where  $1 \leq p \leq d$ .*

For later purpose, let us discuss the particular case  $p = d - 1$ . In this case we describe the branched sphere  $\mathbb{S}_q^d$  as

$$ds^2 = \sin^2 \theta (d\chi^2 + \sin^2 \chi q^2 d\tau^2 + \cos^2 \chi d^2 \Omega_{d-3}) + d\theta^2 , \quad (5.11)$$

with domains

$$\chi \in \left[0, \frac{\pi}{2}\right] , \quad \tau \in [0, 2\pi) , \quad \theta \in [0, \pi] . \quad (5.12)$$

Again by dropping an overall factor  $\sin^2 \theta$  for the metric (5.11), one obtains

$$ds^2 = d\chi^2 + \sin^2 \chi q^2 d\tau^2 + \cos^2 \chi d^2 \Omega_{d-3} + d\eta^2 , \quad (5.13)$$

where  $\cot \theta = \sinh \eta$  and  $\eta \in (-\infty, +\infty)$ . This is the space  $\mathbb{S}_q^{d-1} \times \mathbb{H}^1$ . Here we use  $\mathbb{H}^1$  instead of  $\mathbb{R}^1$  to emphasize that the volume of  $\mathbb{H}^d$  may be regularized. For free fields, one can compute the CFT Rényi entropy on  $\mathbb{S}_q^{d-1} \times \mathbb{H}^1$  and show explicitly that the result agrees with that computed from  $\mathbb{S}_q^d$  or  $\mathbb{S}_q^1 \times \mathbb{H}^{d-1}$ . In consideration of supersymmetry, one has to add a background field  $A_\tau$  along the replica  $\tau$  circle inside  $\mathbb{S}_q^{d-1}$ , in order to find the agreement.

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<sup>27</sup>Again by the universal part of Rényi entropy we refer to the scheme independent part.

## 5.2 Coincidence of backgrounds

Our main concern is physical quantities for CFTs. For this purpose we can work on  $\mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{H}_{1/\sqrt{q}}^1$  instead of  $\mathbb{S}_q^{d-1} \times \mathbb{H}^1$  because they are related by a scale transformation

$$\frac{1}{\sqrt{q}}[\mathbb{S}_q^{d-1} \times \mathbb{H}^1] = [\mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{H}_{1/\sqrt{q}}^1] . \quad (5.14)$$

Furthermore, we focus on the limit  $q \rightarrow 0$ . For this purpose, one can instead consider  $\mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{S}_{1/\sqrt{q}}^1$  because it is equivalent to  $\mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{H}_{1/\sqrt{q}}^1$  in the limit  $q \rightarrow 0$

$$\mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{H}_{1/\sqrt{q}}^1 \Big|_{q \rightarrow 0} = \mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{S}_{1/\sqrt{q}}^1 \Big|_{q \rightarrow 0} . \quad (5.15)$$

In consideration of supersymmetry, one can use the squashed sphere  $\widetilde{\mathbb{S}}_{\sqrt{q}}^{d-1}$  to replace the conic sphere  $\mathbb{S}_{\sqrt{q}}^{d-1}$  in the right hand side of (5.15), because supersymmetric partition functions do not depend on the resolving factor [42, 49, 78–81].<sup>28</sup> (5.15) is useful in the sense that it offers a way to compute the asymptotic supersymmetric Rényi entropy for interacting SCFTs. To do this, one has to make sure that the background gauge field on  $\mathbb{S}_{\sqrt{q}}^{d-1} \times \mathbb{S}_{1/\sqrt{q}}^1$  agrees with that on the original space  $\mathbb{S}_q^d$ . Fortunately we have more knowledge about supersymmetric partition functions on  $\mathbb{S}^{d-1} \times \mathbb{S}^1$  or its generalized version  $\mathbb{S}_b^{d-1} \times \mathbb{S}_\beta^1$ , where  $b$  is the squashing parameter.

## 5.3 Squashed Casimir energy

Now we make a connection between the asymptotic Rényi entropy and Casimir energy. It is known that the partition function  $Z$  on  $\mathbb{S}_b^{d-1} \times \mathbb{S}_\beta^1$  is determined by the Casimir energy on  $\mathbb{S}_b^{d-1}$  in the limit  $\beta \rightarrow \infty$

$$E_c := - \lim_{\beta \rightarrow \infty} \partial_\beta \log Z(\beta) , \quad (5.16)$$

which is equivalent to say

$$\lim_{\beta \rightarrow \infty} \log Z(\beta) = -\beta E_c . \quad (5.17)$$

In this work, we concern the case with supersymmetry. In the particular case of  $6d$   $(2,0)$  theories, the supersymmetric Casimir energy has been studied in [82]<sup>29</sup>, where the authors considered a general 5-sphere with squashing parameters  $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ . The squashing parameters are defined as parameters appearing in the Killing vector

$$K = \omega_1 \frac{\partial}{\partial \phi_1} + \omega_2 \frac{\partial}{\partial \phi_2} + \omega_3 \frac{\partial}{\partial \phi_3} , \quad (5.18)$$

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<sup>28</sup>For this reason, we will not distinguish  $d$ -1-dimensional squashed sphere and conic sphere in the following unless it is necessary.

<sup>29</sup>For the  $6d$   $(2,0)$  superconformal index, see [83–85].

where  $\phi_1, \phi_2, \phi_3$  are three circles representing  $U(1)^3$  isometries of  $\mathbb{S}^5$ . The supersymmetric Casimir energy of an interacting  $(2, 0)$  theory is [82]

$$E_{\mathfrak{g}} = r_{\mathfrak{g}} E_{u(1)} - d_{\mathfrak{g}} h_{\mathfrak{g}}^{\vee} \frac{\sigma_1^2 \sigma_2^2}{24 \omega_1 \omega_2 \omega_3} , \quad (5.19)$$

where  $\sigma_1$  and  $\sigma_2$  are chemical potentials for the two Cartans of the  $SO(5)$  R-symmetry and  $E_{u(1)}$  is given by

$$E_{u(1)} = -\frac{1}{48 \omega_1 \omega_2 \omega_3} \left[ \sigma_1^2 \sigma_2^2 - \sum_{i < j} \omega_i^2 \omega_j^2 + \frac{1}{4} \left( \sum_j \omega_j^2 - \sigma_1^2 - \sigma_2^2 \right)^2 \right] . \quad (5.20)$$

For the particular case of  $\mathbb{S}_q^5 \times \mathbb{S}^1$  (which is equivalent to  $\mathbb{S}_{\sqrt{q}}^5 \times \mathbb{S}_{\frac{1}{\sqrt{q}}}^1$  for CFTs), we should identify the shape parameters as

$$\omega_1 = \omega_2 = 1 , \quad \omega_3 = \frac{1}{q} . \quad (5.21)$$

In the limit  $q \rightarrow 0$ , in order to match our chemical potentials (3.14), we set  $\sigma_1$  and  $\sigma_2$  as <sup>30</sup>

$$\sigma_1^2(q \rightarrow 0) = \frac{r_1^2}{q^2} , \quad \sigma_2^2(q \rightarrow 0) = \frac{r_2^2}{q^2} , \quad \text{with } r_1 + r_2 = 1 . \quad (5.22)$$

Evaluating (5.19) we get

$$E_{\mathfrak{g}} \Big|_{q \rightarrow 0} = -\frac{1}{24} \frac{r_1^2 r_2^2}{q^3} (r_{\mathfrak{g}} + d_{\mathfrak{g}} h_{\mathfrak{g}}^{\vee}) . \quad (5.23)$$

Therefore the free energy <sup>31</sup>

$$f[\mathbb{S}_{q \rightarrow 0}^5 \times \mathbb{S}^1] = \frac{1}{\pi^3} \beta E_{\mathfrak{g}} \Big|_{q \rightarrow 0} = -\frac{1}{12 \pi^2} \frac{r_1^2 r_2^2}{q^3} (r_{\mathfrak{g}} + d_{\mathfrak{g}} h_{\mathfrak{g}}^{\vee}) , \quad (5.24)$$

where we have divided a  $q$ -independent volume factor  $\text{Vol} [\mathbb{D}^4 \times \mathbb{S}^1] = \pi^3$ . Due to (5.15), we have

$$f[\mathbb{S}_{q \rightarrow 0}^5 \times \mathbb{S}^1] = f[\mathbb{S}_{q \rightarrow 0}^1 \times \mathbb{H}^5] , \quad (5.25)$$

from which we obtain the asymptotic supersymmetric Rényi entropy on  $\mathbb{S}_q^1 \times \mathbb{H}^5$

$$S_{q \rightarrow 0} = -I_{q \rightarrow 0} = \frac{1}{12} \frac{r_1^2 r_2^2}{q^3} (r_{\mathfrak{g}} + d_{\mathfrak{g}} h_{\mathfrak{g}}^{\vee}) . \quad (5.26)$$

This fixes the undetermined coefficient  $A$  in (4.1) as

$$A = r_{\mathfrak{g}} + h_{\mathfrak{g}}^{\vee} d_{\mathfrak{g}} . \quad (5.27)$$

Notice that the fact that the free limit of (5.26) precisely agrees with the leading large  $\gamma$  term of (3.15) by itself is nontrivial, which confirms the validity of (5.25) in the free case.

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<sup>30</sup>The  $q$  scalings in chemical potentials appear following the convention in [82].

<sup>31</sup> $f := \frac{I}{V}$ .

## 6. Large $N$ limit

In the large  $N$  limit of the  $(2, 0)$  theory with  $\mathfrak{g} = A_{N-1}$ , the supersymmetric Rényi entropy (4.9) becomes

$$\begin{aligned} \frac{S_\gamma^{(2,0)}}{N^3} &= \frac{1}{12} r_1^2 r_2^2 (\gamma - 1)^3 + \frac{4}{12} r_1 r_2 (\gamma - 1)^2 \\ &\quad + \frac{4}{12} (1 + 2r_1 r_2) (\gamma - 1) + \frac{4}{3} . \end{aligned} \quad (6.1)$$

We will demonstrate in this section that the above large  $N$  result precisely agrees with the holographic result from the seven-dimensional BPS topological black hole in gauged supergravity.

### 6.1 Gauged supergravity

The seven-dimensional gauged  $SO(5)$  supergravity can be obtained by Kaluza-Klein reduction of eleven-dimensional supergravity on  $\mathbb{S}^4$ . For our purpose, we consider a truncation where only the metric, two gauge fields associated to two Cartans of  $SO(5)$  and two scalars are retained. The seven-dimensional Lagrangian is given by [86]

$$\frac{1}{\sqrt{g}} \mathcal{L} = R - \frac{1}{2} (\partial \vec{\phi})^2 - \frac{4}{L^2} V - \frac{1}{4} \sum_{i=1}^2 \frac{1}{X_i^2} \left( F_{(2)}^i \right)^2 , \quad (6.2)$$

where  $\vec{\phi} = (\phi_1, \phi_2)$  are two scalars and

$$X_i = e^{-\frac{1}{2} \vec{a}_i \cdot \vec{\phi}} , i = 1, 2 . \quad \vec{a}_1 = \left( \sqrt{2}, \sqrt{\frac{2}{5}} \right) , \quad \vec{a}_2 = \left( -\sqrt{2}, \sqrt{\frac{2}{5}} \right) . \quad (6.3)$$

The potential is given by

$$V = -4X_1 X_2 - 2X_0 X_1 - 2X_0 X_2 + \frac{1}{2} X_0^2 , \quad X_0 = \frac{1}{X_1 X_2} . \quad (6.4)$$

Note that for two equal scalars and two equal gauge strengths, the Lagrangian (6.2) can be further truncated. Turn to the CFT side, 6d  $(2, 0)$  theories have global  $SO(5)$  R-symmetry, which corresponds to the  $SO(5)$  gauge group in the bulk supergravity. Also there could be two  $U(1)$  background fields used to compensate the singularity on  $\mathbb{S}_q^6$ , which correspond to  $A^1, A^2$  in the gauged supergravity.

### 6.2 Topological black hole

The 2-charge 7d AdS black hole solution for (6.2) was found in [86]

$$\begin{aligned} ds_7^2 &= -\frac{1}{[h_1 h_2]^{\frac{4}{5}}} f(r) dt^2 + [h_1 h_2]^{\frac{1}{5}} \left( \frac{dr^2}{f(r)} + r^2 d\Omega_{5,k}^2 \right) \\ f(r) &= k - \frac{m}{r^4} + \frac{r^2}{L^2} h_1 h_2 , \quad h_i = 1 + \frac{q_i}{r^4} , \end{aligned} \quad (6.5)$$

together with scalars and gauge fields

$$X_i = \frac{[h_1 h_2]^{\frac{2}{5}}}{h_i} , \quad A^i = \left[ \sqrt{k} \left( \frac{1}{h_i} - 1 \right) + \mu_i \right] dt . \quad (6.6)$$

$d\Omega_{5,k}^2$  is the metric on a unit  $\mathbb{S}^5$ ,  $\mathbb{T}^5$  or  $\mathbb{H}^5$  corresponding to  $k = 1, 0, -1$ , respectively. Since our concern is the  $6d$  SCFT on  $\mathbb{S}^1 \times \mathbb{H}^5$ , we are particularly interested in the extremal solution with hyperbolic foliation, where  $m = 0$  and  $k = -1$ . We will first proceed in Lorentz signature and assume a well-defined Wick rotation.

The solution (6.5) is a BPS topological black hole with two charges. For convenience, define a rescaled charge

$$\kappa_i = \frac{q_i}{r_H^4} , \quad (6.7)$$

where the horizon  $r_H$  is the largest root of the equation

$$f(r_H) = 0 . \quad (6.8)$$

Then the horizon can be expressed in terms of  $\kappa_i$

$$r_H = \frac{L}{\sqrt{(1 + \kappa_1)(1 + \kappa_2)}} . \quad (6.9)$$

The Hawking temperature of this black hole is

$$\begin{aligned} T &= \left. \frac{f'(r)}{4\pi\sqrt{h_1 h_2}} \right|_{r=r_H} \\ &= \frac{1 - \kappa_1 - \kappa_2 - 3\kappa_1 \kappa_2}{2\pi L(1 + \kappa_1)(1 + \kappa_2)} . \end{aligned} \quad (6.10)$$

When all charges vanish, we get to the temperature of the uncharged black hole

$$T_0 = \frac{1}{2\pi L} . \quad (6.11)$$

The Bekenstein-Hawking entropy is given by the outer horizon area

$$S = \frac{V_5 L^5}{4G_7} \frac{1}{(1 + \kappa_1)^2 (1 + \kappa_2)^2} , \quad (6.12)$$

where  $G_7$  is the seven dimensional Newton constant and  $V_5$  is the regularized volume of  $\mathbb{H}^5$ . The total charge  $Q_i$  can be computed by Gauss law

$$\begin{aligned} Q_i &= \frac{1}{16\pi G_7} \int_{r \rightarrow \infty} -\sqrt{g} F^{rt} = \frac{V_5}{4\pi G_7} i q_i \\ &= \frac{V_5 L^4}{4\pi G_7} \frac{i \kappa_i}{(1 + \kappa_1)^2 (1 + \kappa_2)^2} . \end{aligned} \quad (6.13)$$

The chemical potential is

$$\mu_i = \frac{i}{\kappa_i^{-1} + 1} . \quad (6.14)$$

### 6.3 Precise check

To match the background gauge fields of the boundary CFT, we set

$$\mu_1 = i(1 - \gamma)\frac{r_1}{2}, \quad \mu_2 = i(1 - \gamma)\frac{r_2}{2}, \quad \text{with } r_1 + r_2 = 1. \quad (6.15)$$

By using these inputs, we can solve  $\kappa_1$  and  $\kappa_2$  by (6.14). Then all physical quantities  $T, S, Q_i$  can be worked out explicitly. One can eventually compute the holographic supersymmetric Rényi entropy using the formula derived in [42]

$$S_q = \frac{q}{1 - q} \int_q^1 \left( \frac{S(n)}{n^2} - \frac{Q_i(n)\mu'_i(n)}{T_0} \right) dn. \quad (6.16)$$

Written in terms of  $\gamma := 1/q$ , the result is given by

$$S_\gamma = \frac{L^5 V_5}{4G_7} \left[ \frac{r_1^2 r_2^2 (\gamma - 1)^3}{16} + \frac{(1 + 2r_1 r_2)(\gamma - 1)}{4} + \frac{(\gamma - 1)^2 r_1 r_2}{4} + 1 \right]. \quad (6.17)$$

By identifying the bulk and boundary parameters,

$$\frac{L^5 V_5}{4G_7} = \frac{4}{3} N^3, \quad (6.18)$$

one can write the holographic result as

$$S_\gamma = N^3 \left( \frac{r_1^2 r_2^2 (\gamma - 1)^3}{12} + \frac{(1 + 2r_1 r_2)(\gamma - 1)}{3} + \frac{(\gamma - 1)^2 r_1 r_2}{3} + \frac{4}{3} \right). \quad (6.19)$$

This precisely agrees with the field theory result (6.1).

## 7. A possible $a/c$ bound

As what has been observed in  $4d$  SCFTs [45], the Rényi entropy inequalities indicate the  $a/c$  bounds in field theories <sup>32</sup>,

$$\partial_q H_q \leq 0, \quad (7.1)$$

$$\partial_q \left( \frac{q-1}{q} H_q \right) \geq 0, \quad (7.2)$$

$$\partial_q ((q-1)H_q) \geq 0, \quad (7.3)$$

$$\partial_q^2 ((q-1)H_q) \leq 0, \quad (7.4)$$

where  $H_q := S_q/S_1$ . Imposing these conditions to our results (4.10)(4.12)(4.15), one obtains

$$0 < \frac{\bar{c}}{\bar{a}} \leq \frac{7}{3}, \quad (7.5)$$

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<sup>32</sup>The validity of these inequalities for supersymmetric Rényi entropy is expected although a proof is still in preparation.

or equivalently

$$\frac{\bar{a}}{\bar{c}} \geq \frac{3}{7} . \quad (7.6)$$

Note that all the  $a, c$  data of the currently known  $6d$   $(2, 0)$  SCFTs, listed in Table 1 in Appendix A, satisfy the inequality (7.5)(7.6). The lowest  $\bar{a}/\bar{c}$  value in the current data,  $4/7$ , supported by the large  $N$  limits, is greater than our bound  $3/7$ . Note that the expression of supersymmetric Rényi entropy in terms of  $a, c$  anomalies could work for theories beyond the ADE type. It would be interesting to understand whether our bound implies new  $(2, 0)$  SCFTs. It would also be interesting to understand similar bounds in SCFTs with less supersymmetry. We leave these questions for future work.

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## A. Data of simply-laced Lie algebra $\mathfrak{g}$

**Table 1:** The rank  $r_{\mathfrak{g}}$ , dual Coxeter number  $h_{\mathfrak{g}}^{\vee}$ , dimension  $d_{\mathfrak{g}}$  of the simply-laced Lie algebras and the normalized  $a, c$  anomalies for the associated  $6d$   $(2, 0)$  SCFTs [22].

$\mathfrak{g}$	$r_{\mathfrak{g}}$	$h_{\mathfrak{g}}^{\vee}$	$d_{\mathfrak{g}}$	$\bar{a}_{\mathfrak{g}}$	$\bar{c}_{\mathfrak{g}}$	$\bar{a}_{\mathfrak{g}}/\bar{c}_{\mathfrak{g}}$
$A_{n-1}$	$n-1$	$n$	$n^2-1$	$\frac{16}{7}n^3 - \frac{9}{7}n - 1$	$4n^3 - 3n - 1$	$\frac{3}{7(2n+1)^2} + \frac{4}{7}$
$D_n$	$n$	$2n-2$	$n(2n-1)$	$\frac{64}{7}n^3 - \frac{96}{7}n^2 + \frac{39}{7}n$	$16n^3 - 24n^2 + 9n$	$\frac{3}{7(3-4n)^2} + \frac{4}{7}$
$E_6$	6	12	78	$\frac{15018}{7}$	3750	$\sim 0.572114$
$E_7$	7	18	133	5479	9583	$\sim 0.571742$
$E_8$	8	30	248	$\frac{119096}{7}$	29768	$\sim 0.571544 > \frac{4}{7}$

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